Solution of Partial Differential Equations

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Basic Concepts :- A partial differential equation (PDE) is an equation involving one or more partial derivatives of an unknown function, u, that depends on two or more variables, often time, t, and one or several variables in space. The order of the highest derivative is called the 'order' of the PDE.

Important Second-Order PDEs.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \longrightarrow \text{One Dimensional Wave Equation}$$

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \rightarrow \text{One Dimensional Heat Equation}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \rightarrow \text{Two Dimensional Laplace Equation}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y) \rightarrow \text{Two Dimensional Poisson Equation}$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right] \rightarrow \text{Two Dimensional Wave Equation}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \rightarrow \text{Three Dimensional Laplace Equation.}$$

Here 'c' is a positive constant, 't' is time, x, y, z are Cartesian Cordinates. Dimension is the number of these coordinates in the equation.

The general linear second order partial differential equation is of the form :-

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G.$$

$$AU_{xx} + BU_{xy} + CU_{yy} + DU_x + EU_y + Fu = G.$$

$A, B, \ldots, G$ are all functions of x and y.

$$\Delta = B^2 - 4AC \rightarrow \text{Discrimnant}.$$

Based on the sign of Δ, the PDEs are classified as:

$\Delta < 0$: Elliptic : $U_{xx} + U_{yy} = 0$
Laplace Equation

$\Delta = 0$: Parabolic : $U_t = U_{xx}$
Heat Conduction Equation

$\Delta > 0$: Hyperbolic : $U_{xx} - U_{yy} = 0$
Wave Equation

In the study of partial differential equations, Usually three types of problems arise :-

(I) Dirichlet's Problem :-

Given a continuous function 'f' on the boundary C of a region R, it is required to find a function $U(x,y)$, satisfying the Laplace equation in R, i.e. to find $u(x,y)$ such that

$$\left.\begin{aligned} U_{xx} + U_{yy} &= 0 \text{ in } R \\ u &= f \text{ on } C \end{aligned}\right]$$

(II) Cauchy's Problem :-

$$\left.\begin{aligned} u_{tt} - u_{xx} &= 0, \quad t>0 \\ u(x,0) &= f(x) \\ \frac{\partial u(x,0)}{\partial t} &= g(x) \end{aligned}\right]$$

(iii) **Neumann's Problem:-**

$$\left.\begin{aligned} u_t &= u_{xx}, \quad t>0 \\ u(x,0) &= f(x) \end{aligned}\right]$$