Differential Equations.

1^{st} order diff. eq's :-

1^{st} order :-

$$\frac{di}{dt} + a_1 i = C$$

n^{th} order :-

$$a_0\frac{d^n i}{dt^n} + a_1\frac{d^{n-1} i}{dt^{n-1}} + \dots + a_{n-1}\frac{di}{dt} + a_n i = C$$

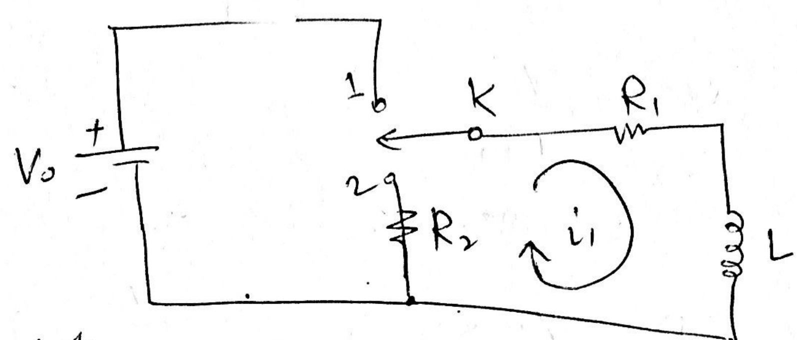
Homogeneous :-

If C is zero, d.eq^n is homogeneous.

If $C \neq 0$, d. eq^n is non-homogeneous.

Linear :- A diff. eq^n is said to be linear if the dependent variable & all its derivatives are of first degree.

R-L Ckt :



After $t=0$, switch is closed from P1 to P2

$$L\frac{di}{dt} + Ri = 0$$

or $$\frac{di}{i} = -\frac{R}{L}dt$$

on integration, $\ln i = -\frac{R}{L}t + \ln k$

$$\ln i = \ln e^{-\frac{Rt}{L}} + \ln k$$

$$\ln i = \ln k \cdot e^{-\frac{Rt}{L}}$$

$$\Rightarrow i = k \cdot e^{-\frac{Rt}{L}}$$

or $$i = k \cdot e^{-t/\tau}$$

$\tau = L/R$ = time constant

→ This is general soln. since k is unknown.

→ To find k, initial conditions have to be taken into account & particular soln. is to be found.

Before switching at $t = 0$,

$$i(0) = \frac{V_0}{R_1}$$

∴ at $t = 0$,

$$\frac{V_0}{R_1} = k \Rightarrow k = \left(\frac{V_0}{R_1}\right)$$

$$\therefore \boxed{i = \frac{V_0}{R_1} e^{-\frac{Rt}{L}}}$$

Soln. to a Non-homogeneous eq using Integrating factor :-

N-H eq be

$$\frac{di}{dt} + Pi = Q$$

where P is a constant

& Q may be either f(t) or const.

$$i = \underbrace{e^{-Pt}\int Q e^{Pt}\,dt}_{\text{particular integral}} + \underbrace{k e^{-Pt}}_{\text{complementary fn.}}$$

---

step voltage response of series R-L ckt :-

Let K be moved fr 2 to 1 at

t = 0;

thus a d.c. step of V_0 volts has been applied to R-L N/w.

→ On applying KVL to N/w, we get

$$L\frac{di}{dt} + Ri = V_0$$

$$\text{or} \quad \frac{di}{dt} + \frac{R}{L}i = \frac{V_0}{L}$$

thus $P = \frac{R}{L}$ & $Q = \frac{V_0}{L}$

$$\therefore \quad i = e^{-Pt} \int Q e^{Pt}\, dt + k e^{-Pt}$$

$$\therefore \quad i = e^{-Rt/L} \int \frac{V_0}{L} e^{\frac{Rt}{L}} + k e^{-Rt/L}$$

$$= e^{-\frac{Rt}{L}} \cdot \frac{V_0}{L} \cdot e^{\frac{Rt}{L}} \cdot \frac{L}{R} + k e^{-Rt/L}$$

$$\boxed{i = \frac{V_0}{R} + k e^{-\frac{Rt}{L}}}$$

→ Applying initial condition that $i = 0$ before & just after ~~closing~~ switching,

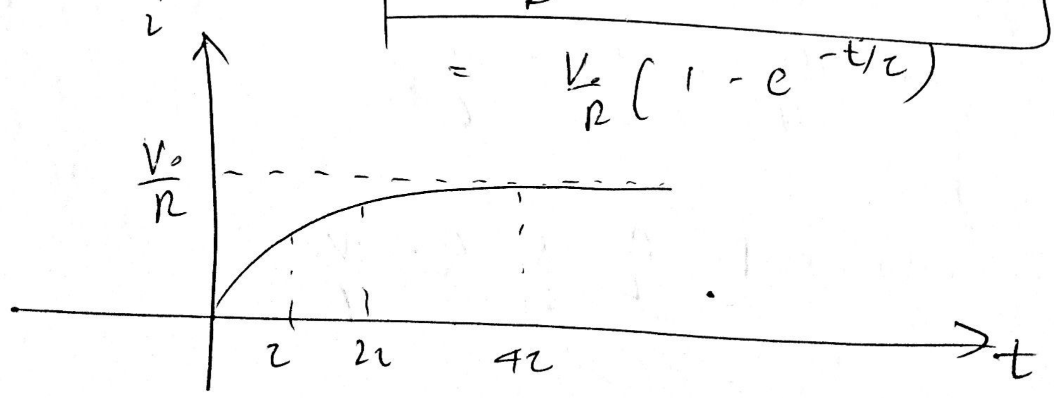
$$0 = \frac{V_0}{R} + k e^{0}$$

$$\Rightarrow \quad k = -\frac{V_0}{R}$$

$$\therefore \quad i = \frac{V_0}{R} - \frac{V_0}{R} e^{-Rt/L}$$

$$\Rightarrow \quad \boxed{i = \frac{V_0}{R}\left(1 - e^{-Rt/L}\right)}$$

$$= \frac{V_0}{R}\left(1 - e^{-t/\tau}\right)$$



Time Constant :-

$i = \frac{V_0}{R} e^{-t/\tau}$

$i = I_0 \, e^{-t/\tau}$

$\therefore \quad \frac{i(t)}{I_0} = e^{-t/\tau}$

At $t = \tau$,

$\frac{i(t)}{I} = e^{-1} = 0.37 = 37\%$

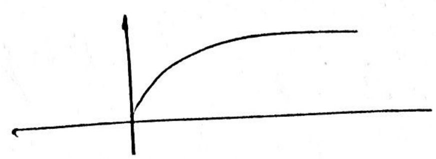
→ gives current reduces to 37% of its initial value after a time interval equal to 1τ.

At $t = 4\tau$,

$\frac{i(t)}{I_0} = e^{-4} = 0.02 = 2\%$

Again, if

$i(t) = I_0 (1 - e^{-t/\tau})$



$\Rightarrow \frac{i(t)}{I_0} = 1 - e^{-t/\tau}$

At $t = \tau$

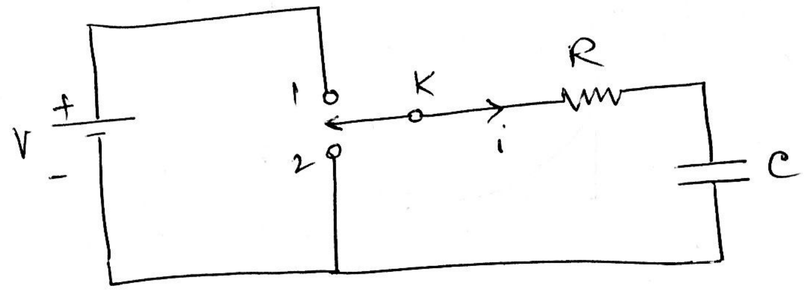
$\frac{i(t)}{I} = 1 - e^{-1} = 1 - \frac{1}{e} = 0.63$

$= 63\%$

$\tau =$ in which I increases to 63%.

Step voltage Response of series R-C Ckt :-

Case I. :-



Case I :- K initially at position 1. then to 2

After switching, KVL gives

$$\frac{1}{c}\int i\,dt + iR = 0$$

$$\Rightarrow R\frac{dq}{dt} + \frac{q}{C} = 0$$

$$\Rightarrow -\frac{1}{RC}\,dt = \frac{dq}{q}$$

Integrating B/S,

$$-\frac{1}{RC}\int dt = \int \frac{dq}{q}$$

$$\text{or} \quad -\frac{t}{RC} = \ln q + \ln k$$

$$\Rightarrow q = ke^{-t/RC}$$

K is evaluated from initial condition

Before switching $q(0) = VC$

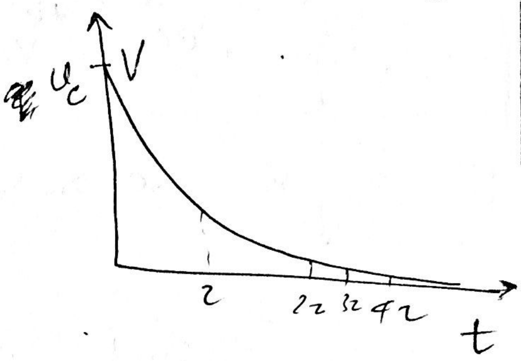
At $t = 0$,

$q = k = VC$

$\therefore \quad q = VC\, e^{-t/RC}$

$$\therefore \quad \boxed{v_c = \frac{q}{C} = V \cdot e^{-t/RC}}$$

$$\boxed{v_c = V e^{-t/\tau}}$$



Case II: Let initial position of switch be 2. Then switched to P1. Now entire voltage V gets applied to RC ckt.

Applying KV to this ckt,

$$iR + \frac{1}{C}\int i\,dt = V$$

$$\Rightarrow \quad R\frac{di}{dt} + \frac{i}{C} = 0$$

$$\text{or} \quad (-RC)\frac{di}{i} = dt$$

On integration -

$$-RC \ln i = t + K$$

K is known from initial conditions. Before switch & after $q_c = 0$ hence after switching $i = V/R$

$$-RC \ln \frac{V}{R} = K$$

$$\therefore \boxed{-RC \ln\left(\frac{V}{R}\right) = k}$$

$$\text{or} \quad -RC \ln i = t - RC \ln \frac{V}{R}$$

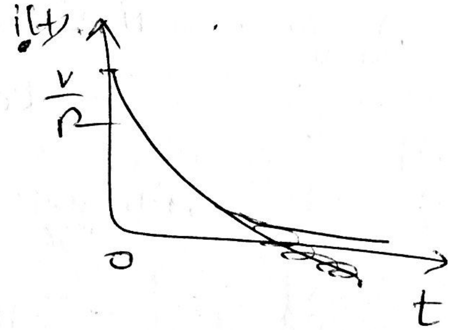
$$\Rightarrow \quad \ln \frac{i}{V/R} = \frac{-t}{RC}$$

$$\text{or} \quad \frac{i}{V/R} = e^{-t/RC}$$

$$\text{or} \quad i = \frac{V}{R} e^{-t/RC}$$

$$\text{or} \quad \boxed{i = I e^{-t/\tau}}$$

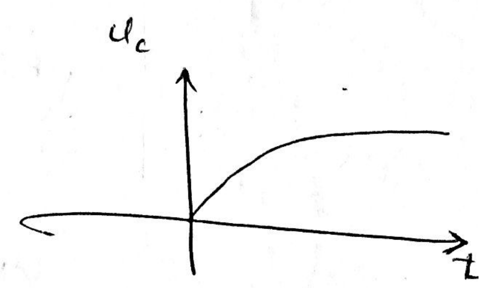
$$\tau = RC$$



$$\boxed{v_R} = IRe^{-t/\tau}$$

$$= Ve^{-t/\tau}$$

$$v_c = \frac{1}{c}\int i\,dt$$

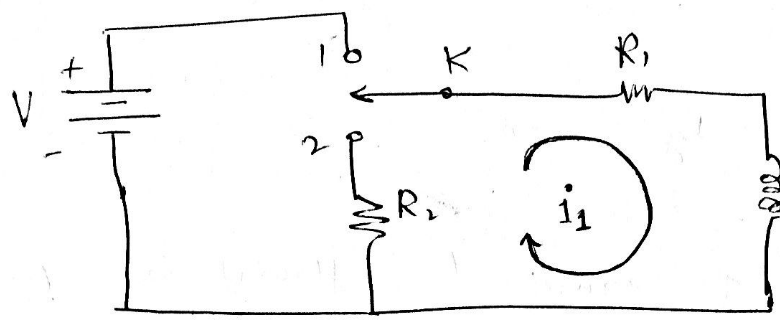


$$\therefore \boxed{v_c} = V - V_R$$

$$= V - Ve^{-t/\tau}$$

$$= V(1 - e^{-t/\tau})$$

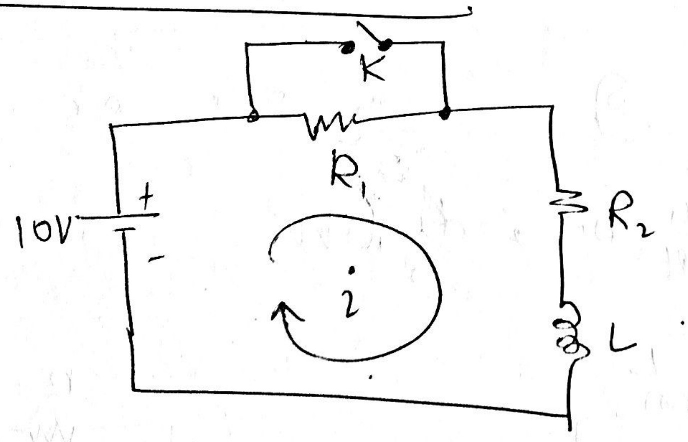
Assignment :- in class



find $i = ?$

$$i = \frac{V}{R_1} e^{-\frac{(R_1+R_2)t}{L}}$$

Assignment for home :-



→ Switch K is closed at time $t = 0$, steady state has reached. obtain expression for current in ckt at any time t.

$$K = \frac{-VR_1}{R_1 + R_2}$$

$$i = \frac{V}{R_2} + K e^{-\frac{R_2 t}{L}}$$

$$i = \frac{V}{R_2}\left[1 - \frac{R_1}{R_1+R_2} e^{-\frac{R_2 t}{L}}\right]$$

Initial Conditions :-

At $t = 0+$,

(i) Replace every L by O.C.

or

a I. gen having I equal to flowing at $t = 0+$

(ii) Replace every C by S.C.

or

a V. gen having src. voltage V_0 equal to $\frac{Q_0}{C}$

if there is an initial charge Q_0.

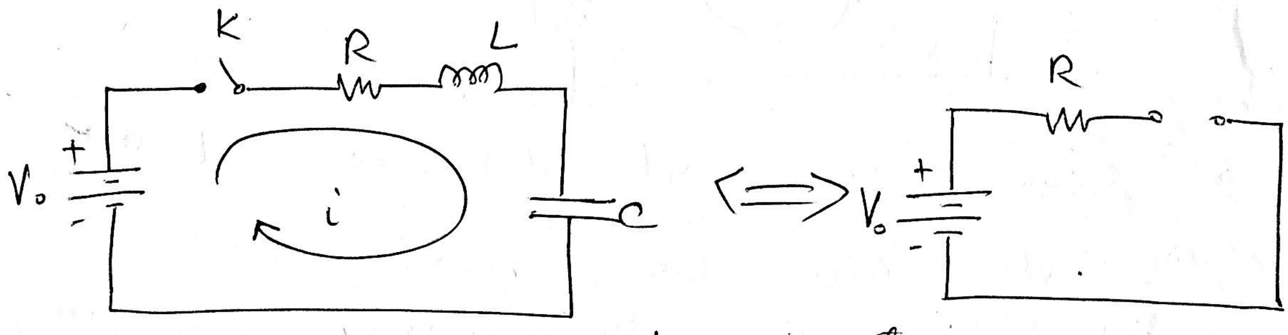
(iii) Leave every R in the N/W unchanged.

from (6)

Initial conditions of a series RLC N/W :-

~~Assignment~~ ~~At Last~~

Find $i(0+)$, $\frac{di}{dt}(0+)$, $\frac{d^2i}{dt^2}(0+)$.



$i(0+) = 0 \longrightarrow$ by observation

$\frac{di}{dt}(0+) = ?$ Applying KVL to N/W,

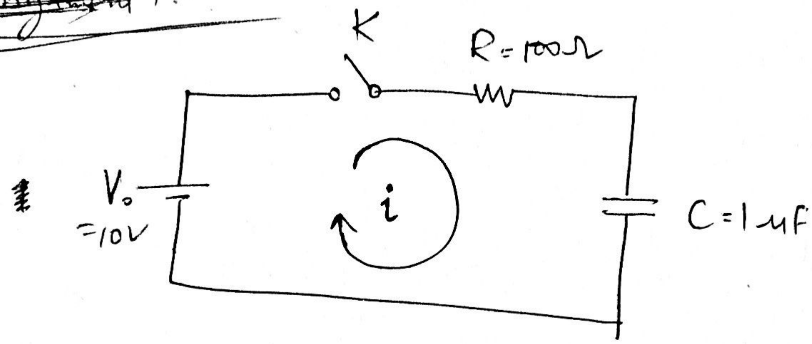
$$L\frac{di}{dt} + iR + \frac{1}{C}\int i\,dt = V_0$$

$$\Rightarrow L\frac{di}{dt} + iR + \frac{q}{C} = V_0$$

But $q(0+) = 0$

& $i(0+) = 0$ $\therefore$ $L\frac{di}{dt}(0+) + 0 + 0 = V_0$ $\Rightarrow$ $\boxed{\frac{di}{dt}(0+) = \frac{V_0}{L}}$

Assignment 1 :-

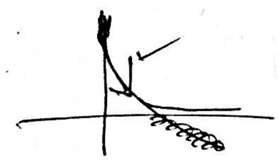


$V_0 = iR + \frac{q}{C}$

$\frac{d}{dt}$

$0 = R\frac{di}{dt} + \frac{i}{C}$

$\Rightarrow \frac{di}{dt} = \frac{-i}{RC}$



find $i(0+)$ & $\frac{di}{dt}(0+)$.

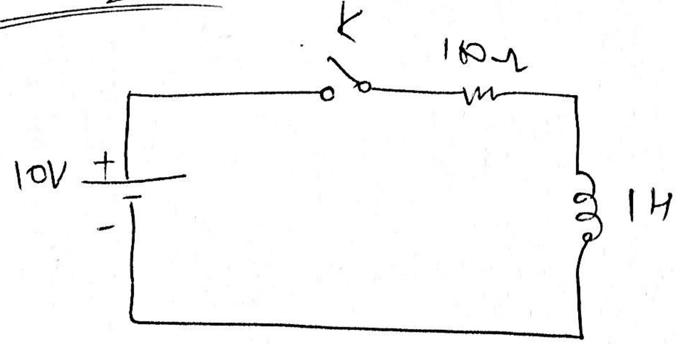
Soln :-

$$V_0 = iR + \frac{1}{C}\int i\,dt$$

$$i(0+) = \frac{V_0}{R} = \frac{10}{100} = 0.1 \text{ A}$$

$$\& \ \frac{di}{dt} = -\frac{1}{CR} \cdot i(0+) = \frac{-0.1}{(10^{-6} \times 100)}$$

Assignment 2 :-



$V_0 = iR + L\frac{di}{dt}$

$\Rightarrow i(0+) = 0$

& $\frac{di}{dt}(0+) = \frac{V_0}{L}$

$= \frac{10}{1}$

$= 10 \ A/s$

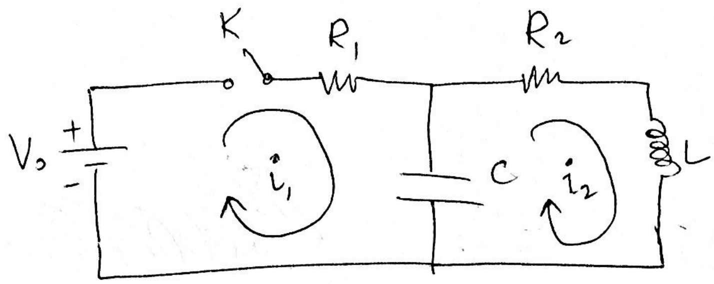
$$i(0+) = 0, \quad \frac{di}{dt}(0+) = 10 \ A/s$$

∴ & $i(0+) = 0$

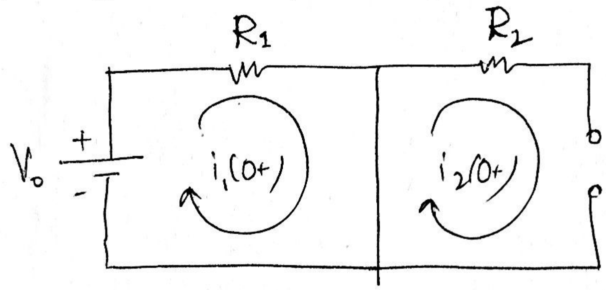
∴ $L \frac{di(0+)}{dt} + 0 + 0 = V_0$

$\Rightarrow \boxed{\frac{di}{dt}(0+) = \frac{V_0}{L}}$

Initial conditions in the case of a 2 mesh RLC ckt :-



At $t = (0+)$, Ckt is :-



Find $i_1(0+)$, $i_2(0+)$, $\frac{di_1}{dt}(0+)$, $\frac{di_2}{dt}(0+)$.

Soln → No initial voltage on C & no initial current thru L.

∴ by inspection, $i_2(0+) = 0$ & $i_1(0+) = \frac{V_0}{R_1}$

Applying KVL to N/W

$$\frac{1}{c}\int (i_1 - i_2)\,dt + R_1 i_1 = V_0 \quad \text{--- (1)}$$

& $\underbrace{+\frac{1}{c}\int (i_2 - i_1)\,dt}_{\text{V rise for } c} - i_2 R_2 - L\frac{di_2}{dt} = 0$

or $$\underbrace{\frac{1}{c}\int (i_1 - i_2)\,dt}_{\text{zero}} + i_2 R_2 + L\frac{di_2}{dt} = 0 \quad \text{--- (2)}$$

In (2), as no charge & $i_2(0+)R_2 = 0$

$$\therefore \frac{di_2(0+)}{dt} = 0$$

In (1), we need $\frac{di_1}{dt}(0+)$

Differentiating (1), we have

$$\frac{(i_1 - i_2)}{c} + R_1\frac{di_1}{dt} = 0$$

At $t = 0+$, $i_2(0+) = 0$, $i_1(0+) = \frac{V_0}{R_1}$

$$\therefore R_1 \frac{di_1}{dt}(0+) = -\frac{1}{c}\left[\frac{V_0}{R_1}\right]$$

or $$\boxed{\frac{di_1(0+)}{dt} = -\frac{V_0}{R_1^2 c}}$$

Now Assignment on 5 Back